On some sampling-related frames in U-invariant spaces

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Abstract

This paper concerns the characterization as frames of some sequences in U-invariant spaces of a separable Hilbert space $\mathcal{H}$ where U denotes an unitary operator defined on $\mathcal{H}$; besides, the dual frames having the same form are also found. This general setting includes, in particular, shift-invariant or modulation-invariant subspaces in $L^2(\mathbb{R})$, where these frames are intimately related to the generalized sampling problem. We also deal with some related perturbation problems. In so doing, we need that the unitary operator U belongs to a continuous group of unitary operators.

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1 Introduction

This paper is concerned with the study of some special frames in U-invariant spaces. Given an unitary operator U on a separable Hilbert space $\mathcal{H}$, we consider closed subspaces having the form $\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a\colon n \in \mathbb{Z}\big\}$, where a denotes some fixed element in $\mathcal{H}$. In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a \,:\, \{\alpha_n\} \in \ell^2(\mathbb{Z}) \Big\}.$$

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Recall that a *Riesz basis* in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator. Any Riesz basis $\{x_n\}_{n\in\mathbb{Z}}$ has a unique biorthogonal (dual) Riesz basis $\{y_n\}_{n\in\mathbb{Z}}$, i.e., $\langle x_n, y_m\rangle_{\mathcal{H}} = \delta_{n,m}$, such that the expansions

$$x = \sum_{n\in\mathbb{Z}} \langle x, y_n\rangle_{\mathcal{H}}\, x_n = \sum_{n\in\mathbb{Z}} \langle x, x_n\rangle_{\mathcal{H}}\, y_n\,,$$

hold for every $x \in \mathcal{H}$. We state the definition by considering the integers set $\mathbb{Z}$ as the index set since throughout the paper most of sequences are indexed in $\mathbb{Z}$. A *Riesz sequence* in $\mathcal{H}$ is a Riesz basis for its closed span. A necessary and sufficient condition in order for the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence in $\mathcal{H}$ is given in Theorem 3 infra.

Given s elements b_j, $j = 1, 2, \ldots, s$, in $\mathcal{A}_a$ a challenging problem is to characterize the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ as a frame (Riesz basis) in $\mathcal{A}_a$, where $r \geq 1$ denotes a positive integer. Besides, another interesting problem is to look for those dual frames having the same form $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ for some $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, so that, for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk} b_j\rangle\, U^{rk} c_j \quad \text{in } \mathcal{H} \tag{1}$$

holds.

At this point, we give an explanation about the expression *sampling-related frames* appearing in the title. Namely, U-invariant subspaces in $\mathcal{H}$ are natural generalizations of shift-invariant or modulation-invariant subspaces of $L^2(\mathbb{R})$; there the unitary involved operators are, respectively, the translation operator $T : f(t) \mapsto f(t-1)$ or the modulation operator $M : f(t) \mapsto \mathrm{e}^{2\pi i t} f(t)$. In the shift-invariant subspace $V_\varphi^2 = \big\{ \sum_n \alpha_n \varphi(t-n) \,:\, \{\alpha_n\} \in \ell^2(\mathbb{Z}) \big\} \subset L^2(\mathbb{R})$ generated by $\varphi \in L^2(\mathbb{R})$, for any $f \in V_\varphi^2$ the inner products $\big\{\langle x, U^{rk} b_j\rangle\big\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ are

$$\int_{-\infty}^{\infty} f(t)\, \overline{b_j(t-rk)}\, dt = \big(f * h_j\big)(rk)\,, \quad k \in \mathbb{Z} \text{ and } j = 1, 2, \ldots, s\,,$$

where $h_j(t) := \overline{b_j(-t)}$ for each $j = 1, 2, \ldots, s$. Thus, the above inner products are nothing but samples of some filtered versions $f * h_j$ of the function f itself: this is precisely the generalized sampling problem in the shift-invariant space V_φ^2. Mathematically, it consists of the stable recovery of any $f \in V_\varphi^2$ from the above sequence of samples, i.e., to obtain sampling formulas in V_φ^2 having the form

$$f(t) = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \big(f * h_j\big)(rk)\, S_j(t-rk)\,, \quad t \in \mathbb{R}\,, \tag{2}$$

such that the sequence of reconstruction functions $\{S_j(\cdot - rk)\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a frame for the shift-invariant space V_φ^2. As a consequence, expansions (1) and (2) have the same nature. Recall that a sequence $\{x_n\}_{n\in\mathbb{Z}}$ is a *frame* for a separable Hilbert space $\mathcal{H}$ if there exist constants $A, B > 0$ (frame bounds) such that

$$A\|x\|^2 \leq \sum_{n\in\mathbb{Z}} |\langle x, x_n\rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in \mathcal{H}\,.$$

A sequence $\{x_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ satisfying only the right hand inequality above is said to be a *Bessel sequence* for $\mathcal{H}$. Given a frame $\{x_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$, the representation property of any vector $x \in \mathcal{H}$ as a series $x = \sum_{n\in\mathbb{Z}} c_n\, x_n$ is retained, but, unlike the case of Riesz bases (*exact frames*), the uniqueness of this representation (for *overcomplete frames*) is sacrificed. Suitable frame coefficients c_n which depend continuously and linearly on x are obtained by using the dual frames $\{y_n\}_{n\in\mathbb{Z}}$ of $\{x_n\}_{n\in\mathbb{Z}}$, i.e., $\{y_n\}_{n\in\mathbb{Z}}$ is another frame for $\mathcal{H}$ such that $x = \sum_{n\in\mathbb{Z}}\langle x, y_n\rangle x_n = \sum_{n\in\mathbb{Z}}\langle x, x_n\rangle y_n$ for each $x \in \mathcal{H}$. For more details on the frame theory see Ref. [7].

Sampling in shift-invariant spaces of $L^2(\mathbb{R})$ has been profusely treated in the mathematical literature (see, for instance, Refs. [4, 5, 8, 9, 11, 15, 20, 24, 25, 26, 27, 28]).

The existence of expansions like (1) in U-invariant subspaces was treated for the first time in [21]; see also [10, 19]. Following similar techniques to those in [21] we give, in Section 2, a complete characterization in $\mathcal{A}_a$ of sequences having the form $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ where $b_j \in \mathcal{A}_a$ for each $j = 1, 2, \dots, s$. In other words, we carry out the study of the completeness, Bessel, frame, or Riesz basis properties of the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$. Whenever it is a frame for $\mathcal{A}_a$ we find a family of dual frames having the same form $\{U^{rk}c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ for some $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$. In Section 4 we also discuss the case where some $b_j \notin \mathcal{A}_a$; although the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is not contained in $\mathcal{A}_a$, something can be said in the light of the theory of pseudo frames (see [17, 18]).

All the obtained results in Section 2 involve the discrete group of unitary operators $\{U^n\}_{n\in\mathbb{Z}}$ which is completely determined by U. If we want to deal with something similar to the time-jitter error version of (2), i.e., the recovery of any $f \in V_\varphi^2$ from a perturbed sequence of samples $\{(f * h_j)(rk+\epsilon_{kj})\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ with errors $\epsilon_{kj} \in \mathbb{R}$, then the availability of a continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ containing, in particular, the operator U (say for instance $U = U^1$) becomes essential. In Section 3, after a brief on groups of unitary operators, we deal with two types of perturbation problems. The first one concerns the study of sequences as $\{U^{rk+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ in $\mathcal{H}$: for small enough error sequence $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}$ ℓ^2-norm, the sequence is a Riesz sequence in $\mathcal{H}$. The second one goes into the recovery of any $x \in \mathcal{A}_a$ from the perturbed sequence of inner products $\{\langle x, U^{rk+\epsilon_{kj}}b_j\rangle_{\mathcal{H}}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$: for small enough errors ϵ_{mj}, there exists a frame expansion for $x \in \mathcal{A}_a$ having the inner products $\{\langle x, U^{rk+\epsilon_{kj}}b_j\rangle_{\mathcal{H}}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ as coefficients.

2 U-invariant subspaces

In a Hilbert space $\mathcal{H}$, the U-invariant subspaces are intimately related to stationary sequences:

2.1 Some preliminaries on stationary sequences

A sequence $\mathbf{s} = \{s_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is said to be *stationary* if

$$\langle s_{m+k}, s_{n+k}\rangle_{\mathcal{H}} = \langle s_m, s_n\rangle_{\mathcal{H}} \quad \text{for all } \ m, n, k \in \mathbb{Z}\,.$$

The function $R_{\mathbf{s}}(k) := \langle s_k, s_0 \rangle_{\mathcal{H}}$, for every $k \in \mathbb{Z}$, is called the *auto-covariance* function of the sequence $\mathbf{s}$. Moreover, two stationary sequences $\mathbf{s} = \{s_k\}_{k\in\mathbb{Z}}$ and $\mathbf{w} = \{w_k\}_{k\in\mathbb{Z}}$ are said to be *stationary correlated* if

$$\langle s_{m+k}, w_{n+k} \rangle_{\mathcal{H}} = \langle s_m, w_n \rangle_{\mathcal{H}} \quad \text{for all } \; m, n, k \in \mathbb{Z},$$

and $R_{\mathbf{s},\mathbf{w}}(k) := \langle s_k, w_0 \rangle_{\mathcal{H}}$, for every $k \in \mathbb{Z}$ defines the corresponding *cross-covariance* function. The following result is a well-known characterization of stationary sequences (see [16]):

Lemma 1. *To every stationary sequence $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ there exists a unique unitary operator $U : \mathcal{H} \to \mathcal{H}$ and $s \in \mathcal{H}$ such that $s_n = U^n s$ for all $n \in \mathbb{Z}$. Conversely every pair (U, s) of a unitary operator U and an $s \in \mathcal{H}$ defines by $s_n = U^n s$, $n \in \mathbb{Z}$, a stationary sequence $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$.*
Moreover, two stationary sequence $\mathbf{s}$ and $\mathbf{w}$ are stationary correlated if and only if they are generated by the same unitary operator U, i.e., $s_n = U^n s$ and $w_n = U^n w$ for some $s, w \in \mathcal{H}$.

The auto-covariance $R_{\mathbf{s}}$ and the cross-covariance $R_{\mathbf{s},\mathbf{w}}$ functions admit a spectral representation which is related to the integral representation of the unitary operator U (see [16]):

Theorem 2. *For every stationary sequence $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ the auto-covariance function $R_{\mathbf{s}}$ admits a spectral representation*

$$R_{\mathbf{s}}(k) = \langle s_k, s_0 \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} \, d\mu_{\mathbf{s}}(\theta) \,, \quad k \in \mathbb{Z} \,, \tag{3}$$

in the form of an integral with respect to a (positive) spectral measure $\mu_{\mathbf{s}}$.
For every two stationary correlated sequences $\mathbf{s} = \{s_n\}_{n\in\mathbb{Z}}$, $\mathbf{w} = \{w_n\}_{n\in\mathbb{Z}}$ in a Hilbert space $\mathcal{H}$ the cross-covariance function admits a spectral representation

$$R_{\mathbf{s},\mathbf{w}}(k) = \langle s_k, w_0 \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\theta} \, d\mu_{\mathbf{s},\mathbf{w}}(\theta) \,, \quad k \in \mathbb{Z} \,, \tag{4}$$

in the form of an integral with respect to a (complex) spectral measure $\mu_{\mathbf{s},\mathbf{w}}$.

2.1.1 The U-invariant subspace $\mathcal{A}_a$

For a fixed $a \in \mathcal{H}$, we consider the subspace of $\mathcal{H}$ given by $\mathcal{A}_a := \overline{\operatorname{span}}\big\{U^n a, \; n \in \mathbb{Z}\big\}$. In case that the sequence $\{U^n a\}_{n\in\mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ we have

$$\mathcal{A}_a = \Big\{ \sum_{n\in\mathbb{Z}} \alpha_n U^n a : \, \{\alpha_n\} \in \ell^2(\mathbb{Z}) \Big\} \,.$$

A necessary and sufficient condition in order for the sequence $\mathbf{a} = \{U^n a\}_{n\in\mathbb{Z}}$ to be a Riesz sequence in $\mathcal{H}$ can be stated in terms of the Lebesgue decomposition of the spectral measure $\mu_{\mathbf{a}}$ into an absolute and a singular part as $d\mu_{\mathbf{a}}(\theta) = \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta + d\mu^s_{\mathbf{a}}(\theta)$:

Theorem 3. *Let* $\mathbf{a} := \{U^n a\}_{n\in\mathbb{Z}}$ *be a sequence obtained from an unitary operator in a separable Hilbert space* $\mathcal{H}$ *with spectral measure* $d\mu_{\mathbf{a}}(\theta) = \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta + d\mu^s_{\mathbf{a}}(\theta)$*, and let* $\mathcal{A}_a$ *be the closed subspace spanned by* $\mathbf{a}$*. Then* $\mathbf{a}$ *is a Riesz basis for* $\mathcal{A}_a$ *if and only if* $\mu^s_{\mathbf{a}} \equiv 0$ *and*

$$0 < A_{\mathbf{a}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \phi_{\mathbf{a}}(\zeta) \le B_{\mathbf{a}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \phi_{\mathbf{a}}(\zeta) < \infty\,, \tag{5}$$

where $\mathbb{T} := \{\mathrm{e}^{\mathrm{i}\theta}\,:\,\theta\in[-\pi,\pi)\}$*.*

Proof. For a fixed ℓ^2-sequence $c := \{c_n\}_{n\in\mathbb{Z}}$ we have

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}} c_k U^k a\Big\|^2 &= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_m\overline{c_n}\langle U^m a, U^n a\rangle = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_m\overline{c_n}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}(m-n)\theta}d\mu_{\mathbf{a}}(\theta)\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} c_m\overline{c_n}\mathrm{e}^{\mathrm{i}(m-n)\theta}d\mu_{\mathbf{a}}(\theta)\\ &=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2 d\mu_{\mathbf{a}}(\theta)\,,\end{aligned} \tag{6}$$

if $\mu_{\mathbf{a}}$ is not absolutely continuous with respect to Lebesgue measure λ, Lemma 4 below implies that there exists a bounded sequence $\{c^N\}_{N=1}^{\infty} \subset \ell^2(\mathbb{Z})$ such that $\big\|\sum_{k\in\mathbb{Z}} c^N_k U^k a\big\|^2$ tends to infinity with N, so $\mathbf{a}$ cannot be a Bessel sequence, therefore, not a Riesz basis. Assume now that $\mu^s_{\mathbf{a}} \equiv 0$ and $\mathbf{a}$ is a Riesz basis, this implies that there exists $B < \infty$ such that,

$$\Big\|\sum_{k\in\mathbb{Z}} c_k U^k a\Big\|^2 \le B\|c\|^2_{\ell^2} \tag{7}$$

for all $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Let $\Omega_n := \{\zeta\in\mathbb{T}\,:\,\phi_{\mathbf{a}}(\zeta) \ge n\}$, assume $B_{\mathbf{a}} = \infty$, this implies that $\lambda(\Omega_n) > 0$ for all n. Now, (6), (7) and Parseval's theorem implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta \le \frac{B}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2 d\theta \tag{8}$$

for all $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Introducing the Fourier expansion of the characteristic function $\chi_{\Omega_n} \in L^2(-\pi,\pi)$ in (8) we obtain $n \le B$ for all n, which contradicts $B < \infty$; thus the assumption $B_{\mathbf{a}} = \infty$ is false. In a similar way it can be proved that $0 < A_{\mathbf{a}}$.

For the sufficient condition, assume now that $\mu^s_{\mathbf{a}} \equiv 0$, then (6) implies

$$\Big\|\sum_{k\in\mathbb{Z}} c_k U^k a\Big\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta})d\theta\,,$$

if, in addition, condition (5) is satisfied we get

$$\frac{A_{\mathbf{a}}}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2 d\theta \le \Big\|\sum_{k\in\mathbb{Z}} c_k U^k a\Big\|^2 \le \frac{B_{\mathbf{a}}}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{k\in\mathbb{Z}} c_k\mathrm{e}^{\mathrm{i}k\theta}\Big|^2 d\theta\,,$$

which implies

$$A_{\mathbf{a}}\|c\|^2_{\ell^2} \le \Big\|\sum_{k\in\mathbb{Z}} c_k U^k a\Big\|^2 \le B_{\mathbf{a}}\|c\|^2_{\ell^2} \quad \text{for each } c\in\ell^2(\mathbb{Z})\,.$$

Therefore, the sequence $\mathbf{a}$ is a Riesz basis for $\mathcal{A}_a$. $\square$

Lemma 4. *Let μ be a finite positive measure on $(-\pi,\pi)$ which is not absolutely continuous with respect to Lebesgue measure λ. Then, there exists a bounded sequence $\{c^N\}_{N=1}^{\infty} \subset \ell^2(\mathbb{Z})$ such that*

$$\lim_{N\to\infty} \Big\| \sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta} \Big\|^2_{L^2_\mu(-\pi,\pi)} = \infty \,.$$

Proof. If the measure μ is not absolutely continuous with respect to Lebesgue measure, then $\mu(M) > 0$ for a Lebesgue measurable set $M \subset (-\pi,\pi)$ of Lebesgue measure zero. Thus, there exists a Borel set B, in fact an intersection of a countable collection of open sets, of Lebesgue measure zero such that $M \subset B \subset (-\pi,\pi)$ (see [22, p. 63]). Therefore, $\mu(B) > 0$ and $\lambda(B) = 0$. On the other hand, every finite Borel measure on $(-\pi,\pi)$ is inner regular (see [22, p. 340]), that is,

$$\mu(B) = \sup\big\{\mu(C)\colon C \subset B,\, C \text{ compact}\big\}\,,$$

then there exists a compact set $C \subset (-\pi,\pi)$ such that

$$\mu(C) > 0 \quad \text{and} \quad \lambda(C) = 0\,.$$

For any $\varepsilon > 0$ there exists a sequence of disjoint open intervals $I_j \subset (-\pi,\pi)$ such that

$$C \subset \bigcup_{j=1}^{\infty} I_j \quad \text{and} \quad \sum_{j=1}^{\infty} \lambda(I_j) \le \lambda(C) + \varepsilon = \varepsilon\,,$$

(see [22, pp. 58 and 42]). Since C is compact we may take the sequence to be finite. Hence, for every $N \in \mathbb{N}$ there exist open disjoint intervals $I_1^N, I_2^N, \ldots, I_{i_N}^N$ in $(-\pi,\pi)$ such that

$$C \subset \bigcup_{i=1}^{i_N} I_i^N \quad \text{and} \quad \sum_{i=1}^{i_N} \lambda(I_i^N) \le \frac{1}{3^N}\,.$$

Besides, $\sum_{i=1}^{i_N} \mu(I_i^N) \ge \mu(C)$.

Consider the function $g_N\colon (-\pi,\pi) \to \mathbb{R}$, where $g_N = 2^{N/2}\chi_{\bigcup_{i=1}^{i_N} I_i^N}$, that satisfies

$$\|g_N\|_2^2 = 2^N \sum_{i=1}^{i_N} \lambda(I_i^N) \le \frac{2^N}{3^N} < 1\,.$$

We modify and extend each g_N to obtain a 2π-periodic function $f_N\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that f_N and its derivative are continuous on $\mathbb{R}$, $\|f_N\|_2^2 \le 1$ and $f_N(\theta) = g_N(\theta)$ for every $\theta \in \bigcup_{i=1}^{i_N} I_i^N$. Let $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ be the Fourier series of f_N. First, by using Parseval's identity we have

$$\|c_k^N\|_{\ell^2}^2 = \frac{1}{2\pi}\|f_N\|_2^2 \le \frac{1}{2\pi} \quad \text{for every } N \in \mathbb{N}\,,$$

so that $\{c^N\}_{N=1}^{\infty}$ is a bounded sequence in $\ell^2(\mathbb{Z})$. Besides, the regularity of each f_N ensures that each Fourier series converges uniformly to f_N. Therefore each series $\sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta}$ converges to f_N in $L^2_\mu(-\pi,\pi)$ and consequently,

$$\Big\| \sum_k c_k^N \mathrm{e}^{\mathrm{i}k\theta} \Big\|^2_{L^2_\mu(-\pi,\pi)} = \int_{-\pi}^{\pi} |f_N|^2 d\mu \ge \int_{-\pi}^{\pi} |g_N|^2 d\mu = 2^N \sum_{i=1}^{i_N} \mu(I_i^N) \ge 2^N \mu(C)\,,$$

from which we obtain the desired result. □

The proof of Theorem 3 is similar to that of Theorem 6 in [19], except we do not exclude the case in which the singular measure is atomless. Recently, we have been awared of that Theorem 3 was exposed in [12].

2.2 Studying the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ in $\mathcal{A}_a$

For $b_j \in \mathcal{A}_a,\ j = 1,2,\dots s$, consider the sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$. For every $j = 1,2,\dots s$, the spectral measure $\mu_{\mathbf{a},\mathbf{b_j}}$ in the integral representation of the cross-covariance function of the sequences $\mathbf{a} := \{U^k a\}_{k\in\mathbb{Z}}$ and $\mathbf{b_j} := \{U^k b_j\}_{k\in\mathbb{Z}}$ has no singular part. Indeed, according to Theorem 3, the spectral measure associated with the auto-covariance function of the sequence $\{U^k a\}_{k\in\mathbb{Z}}$ has no singular part; then by using the Cauchy-Schwarz type inequality in [6, p. 125] we get the result. In the sequel we will use the abridged notation $b_{k,j} := U^{rk}b_j$; our goal in this section is to study the sequence $\{b_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ in $\mathcal{A}_a$ in terms of an $s\times r$ matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})$ introduced below. For the sake of completeness we include some needed calculations which appear in [21].

First of all, we have

$$\langle U^k a, b_{n,j}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-rn)\theta}\phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta})d\theta\,,$$

where $\phi_{\mathbf{a},\mathbf{b}_j}$ stands for the cross spectral density of the stationary correlated sequences $\mathbf{a} := \{U^k a\}_{k\in\mathbb{Z}}$ and $\mathbf{b}_j := \{U^k b_j\}_{k\in\mathbb{Z}}$. Define

$$\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(\phi_{\mathbf{a},\mathbf{b}_1}(\mathrm{e}^{\mathrm{i}\theta}), \phi_{\mathbf{a},\mathbf{b}_2}(\mathrm{e}^{\mathrm{i}\theta}),\dots,\phi_{\mathbf{a},\mathbf{b}_s}(\mathrm{e}^{\mathrm{i}\theta})\big)^\top .$$

In what follows we will use the *left-shift operator* S defined as

$$\begin{array}{rccc} S: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{k+1}\mathrm{e}^{\mathrm{i}k\theta}\,, \end{array}$$

or equivalently, by $(Sf)(\mathrm{e}^{\mathrm{i}\theta}) = f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}\theta}$, where $\mathbb{T} := \{\mathrm{e}^{\mathrm{i}\theta} : \theta\in[-\pi,\pi)\}$ denotes the unidimensional torus. Also, we will consider the *decimation operator* D_r, r a positive integer, defined as

$$\begin{array}{rccc} D_r: & L^2(\mathbb{T}) & \longrightarrow & L^2(\mathbb{T}) \\ & \sum_{k\in\mathbb{Z}} a_k \mathrm{e}^{\mathrm{i}k\theta} & \longmapsto & \sum_{k\in\mathbb{Z}} a_{rk}\mathrm{e}^{\mathrm{i}k\theta}\,, \end{array}$$

which can equivalently be written as

$$(D_r f)(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{r}\sum_{k=0}^{r-1} f(\mathrm{e}^{\mathrm{i}\frac{\theta+2k\pi}{r}})\,.$$

For each $l = 0,1,\dots,r-1$, set the $s\times 1$ matrix of functions on the torus $\mathbb{T}$

$$\mathbf{\Psi}^l_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(D_r S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}\big)(\mathrm{e}^{\mathrm{i}\theta})\,,$$

and define the $s\times r$ matrix of functions on the torus $\mathbb{T}$

$$\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := \Big(\mathbf{\Psi}^0_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\ \mathbf{\Psi}^1_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\ \dots\dots\ \mathbf{\Psi}^{r-1}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\Big)\,. \tag{9}$$

It is worth to mention that the matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}$ was explicitly computed in [21] for the translation and modulation cases in $L^2(\mathbb{R})$.

Next, for any $x \in \mathcal{A}_a$, we obtain an expression for the inner products $\alpha_{n,j} := \langle x, U^{rn} b_j\rangle$, $n \in \mathbb{Z}$ and $j = 1, 2, \ldots, s$. Indeed, writing $x = \sum_{k\in\mathbb{Z}} x_k U^k a$ where $\{x_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have:

$$\begin{aligned}\alpha_{n,j} =&\langle x, U^{rn} b_j\rangle = \sum_{k\in\mathbb{Z}} x_k \langle U^k a, U^{rn} b_j\rangle = \sum_{k\in\mathbb{Z}} x_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-rn)\theta}\phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta})d\theta \\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k\in\mathbb{Z}} x_k \mathrm{e}^{\mathrm{i}k\theta}\phi_{\mathbf{a},\mathbf{b}_j}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}rn\theta}d\theta\,,\end{aligned}$$

that is,

$$\boldsymbol{\alpha}_n := \big(\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,s}\big)^\top = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})X(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}rn\theta}d\theta\,, \tag{10}$$

where $X(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}} x_k \mathrm{e}^{\mathrm{i}k\theta}$.

Now, for $l = 0, 1, \ldots, r-1$, define the sequence $x^{(l)} := \{x_k^{(l)} := x_{kr+l}\}_{k\in\mathbb{Z}}$. Thus, we can write

$$X(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{l=0}^{r-1}\sum_{k\in\mathbb{Z}} x_{kr+l}\mathrm{e}^{\mathrm{i}(kr+l)\theta} = \sum_{l=0}^{r-1} X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta})\mathrm{e}^{\mathrm{i}l\theta}\,, \tag{11}$$

where $X^{(l)}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k\in\mathbb{Z}} x_k^{(l)}\mathrm{e}^{\mathrm{i}k\theta}$.

Using Eq. (11) in Eq. (10), we obtain

$$\boldsymbol{\alpha}_n = \sum_{l=0}^{r-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta})\mathrm{e}^{\mathrm{i}l\theta}\mathrm{e}^{-\mathrm{i}rn\theta}d\theta\,.$$

After some easy calculations we get

$$\begin{aligned}\boldsymbol{\alpha}_n &= \sum_{l=0}^{r-1}\frac{1}{2\pi}\int_{-\pi}^{\pi} S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})X^{(l)}(\mathrm{e}^{\mathrm{i}r\theta})\mathrm{e}^{-\mathrm{i}rn\theta}d\theta \\ &= \sum_{l=0}^{r-1}\frac{1}{2\pi}\int_{-r\pi}^{r\pi}\frac{S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{r}})}{r}X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta \\ &= \sum_{l=0}^{r-1}\sum_{k=0}^{r-1}\int_{2\pi k}^{2\pi(k+1)}\frac{S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta}{r}})}{2\pi r}X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta \\ &= \int_0^{2\pi}\sum_{l=0}^{r-1}\sum_{k=0}^{r-1}\frac{S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\frac{\theta+2\pi k}{r}})}{2\pi r}X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{l=0}^{r-1}(D_r S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}})(\mathrm{e}^{\mathrm{i}\theta})X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta}d\theta\,.\end{aligned} \tag{12}$$

Defining $\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_k\,\mathrm{e}^{\mathrm{i}k\theta}$, Eq. (12) implies that

$$\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{l=0}^{r-1}\big(D_r S^{-l}\mathbf{\Phi}_{\mathbf{a},\mathbf{b}}\big)(\mathrm{e}^{\mathrm{i}\theta})X^{(l)}(\mathrm{e}^{\mathrm{i}\theta})\,,$$

which can be written in matrix form as,

$$\begin{aligned}\mathbf{C}(\mathrm{e}^{\mathrm{i}\theta}) &= \Big(\sum_{k\in\mathbb{Z}} \alpha_{k,1}\,\mathrm{e}^{\mathrm{i}k\theta}, \sum_{k\in\mathbb{Z}} \alpha_{k,2}\,\mathrm{e}^{\mathrm{i}k\theta}, \dots, \sum_{k\in\mathbb{Z}} \alpha_{k,s}\,\mathrm{e}^{\mathrm{i}k\theta}\Big)^\top \\ &= \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\big(X^{(0)}(\mathrm{e}^{\mathrm{i}\theta}), X^{(1)}(\mathrm{e}^{\mathrm{i}\theta}), \dots, X^{(r-1)}(\mathrm{e}^{\mathrm{i}\theta})\big)^\top = \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}) \\ &= (L_{\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}}\widetilde{\mathbf{X}})(\mathrm{e}^{\mathrm{i}\theta})\end{aligned} \tag{13}$$

where $L_{\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}} : L^2_r(\mathbb{T}) \longrightarrow L^2_s(\mathbb{T})$ denotes the multiplication operator by $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}$ and

$$\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(X^{(0)}(\mathrm{e}^{\mathrm{i}\theta}), X^{(1)}(\mathrm{e}^{\mathrm{i}\theta}), \dots, X^{(r-1)}(\mathrm{e}^{\mathrm{i}\theta})\big)^\top. \tag{14}$$

By $L^2_r(\mathbb{T})$ (respectively $L^2_s(\mathbb{T})$) we denote the product space $L^2(\mathbb{T})\times\cdots\times L^2(\mathbb{T})$ r times (respectively s times). Besides,

$$\begin{aligned}\|\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}), \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\rangle_{\mathbb{C}^r}\, d\theta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle \mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta}), \widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\rangle_{\mathbb{C}^r}\, d\theta\,.\end{aligned} \tag{15}$$

The above calculations let us prove the following result:

Theorem 5. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \dots, s$ and let $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}$ be the associated matrix given in* (9)*. Then, the following results hold:*

(a) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a complete system in $\mathcal{A}_a$ if and only if the rank of the matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ is r a.e. ζ in $\mathbb{T}$.

(b) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only if there exists a constant $B < \infty$ such that

$$\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) \le B\,\mathbb{I}_r \qquad a.e.\ \zeta \text{ in } \mathbb{T}. \tag{16}$$

(c) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$ if and only if there exist constants $0 < A \le B < \infty$ such that

$$A\,\mathbb{I}_r \le \mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) \le B\,\mathbb{I}_r \qquad a.e.\ \zeta \text{ in } \mathbb{T}. \tag{17}$$

(d) The sequence $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a Riesz basis for $\mathcal{A}_a$ if and only if it is a frame and $s = r$.

Proof. To prove (a), assume that there exists a set $\Omega \subseteq \mathbb{T}$ with positive measure such that rank $[\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)] < r$ for each $\zeta \in \Omega$. Then, there exists a measurable function $v(\zeta)$, $\zeta \in \Omega$, such that $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)v(\zeta) = 0$ and $\|v(\zeta)\|_{L^2_r(\mathbb{T})} = 1$ in Ω. This function can be constructed as in [14, Lemma 2.4]. Define $\widetilde{\mathbf{V}} \in L^2_r(\mathbb{T})$ such that $\widetilde{\mathbf{V}}(\zeta) = v(\zeta)$ if $\zeta \in \Omega$, and $\widetilde{\mathbf{V}}(\zeta) = 0$ if $\zeta \in \mathbb{T}\setminus\Omega$. Hence, from (13) we obtain that the system is not complete. Conversely, if the system is not complete, by using (13) we obtain a $\widetilde{\mathbf{V}}(\zeta)$ different from 0 in a set with positive measure such that $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\widetilde{\mathbf{V}}(\zeta) = 0$. Thus rank $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta) < r$ on a set with positive measure.

To prove (b), we keep in mind that $\{U^k a\}_{k\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{A}_a$, the mapping $T:\ell^2(\mathbb{Z})\to\mathcal{A}_a$, given by $T\{x_k\}_{k\in\mathbb{Z}}=x=\sum_{k\in\mathbb{Z}}x_kU^ka$ is bijective and there exist two constants $0<m_a\le M_a<\infty$ such that

$$m_a\|\{x_k\}\|^2_{\ell^2}\le\|T\{x_k\}\|^2_{\mathcal{H}}\le M_a\|\{x_k\}\|^2_{\ell^2}\,. \tag{18}$$

Assume first that (16) is satisfied. It follows from (13) and (15) that

$$\|\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}\le B\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}\,. \tag{19}$$

By construction $\|\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}=\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2$ and $\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}=\|\{x_k\}_{k\in\mathbb{Z}}\|^2_{\ell^2}$. Using (18), it follows from (19) that

$$\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2\le\frac{B}{m_a}\|x\|^2_{\mathcal{H}}$$

Conversely, assume that $\{b_{kj}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Bessel sequence for $\mathcal{A}_a$, then there exists $0<B'<\infty$ such that

$$\sum_{j=1}^s\sum_{k\in\mathbb{Z}}|\langle x,b_{k,j}\rangle|^2\le B'\|x\|^2_{\mathcal{H}}\,.$$

Using (18), this implies

$$\|\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}\le B'\,M_a\|\widetilde{\mathbf{X}}\|^2_{L^2_r(\mathbb{T})}$$

for all $\widetilde{\mathbf{X}}\in L^2_r(\mathbb{T})$. Inserting the right hand side of (15) for $\|\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}\widetilde{\mathbf{X}}\|^2_{L^2_s(\mathbb{T})}$, it is straightforward to see that (16) holds with $B=B'\,M_a$. The proof of (c) is completed proceeding as in (b).
To prove (d) consider the mapping

$$\begin{aligned}S:\mathcal{A}_a&\longrightarrow\ell^2_s(\mathbb{Z})\\ x&\longmapsto\{\langle x,b_{k,j}\rangle\}_{k\in\mathbb{Z};\,j=1,2,\dots s}\,.\end{aligned}$$

According to (13), the mapping S is isometric equivalent to $L_{\Psi_{a,\mathbf{b}}}$, and assuming that $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame, it is a Riesz basis if and only if S is surjective.
First, if $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Riesz basis, then it is a frame and S is surjective. Applying (a) yields that $L_{\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}}$ is bijective, and therefore $L^*_{\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}}=L_{\Psi^*_{\mathbf{a},\mathbf{b}}}$ is bijective. Hence, $\operatorname{rank}[\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)]$ is s for almost every ζ in $\mathbb{T}$ so

$$r=\operatorname{rank}[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)]=\operatorname{rank}[\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)]=s,$$

and finally $s=r$.
Conversely, if $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame and $s=r$, (a) implies that $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ is invertible for almost every ζ in $\mathbb{T}$, which implies that $L_{\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}}$ is surjective, then S is surjective and $\{b_{k,j}\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Riesz basis. □

The following lemma will allow us to restate Theorem 5:

Lemma 6. *Let $\mathbf{G}(\zeta)$ be an $s\times r$ matrix with entries in $L^2(\mathbb{T})$, and consider the constants*

$$A_{\mathbf{G}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \ \lambda_{\min}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big] \quad \textit{and} \quad B_{\mathbf{G}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \ \lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]\,,$$

where $\lambda_{\min}$ (respectively $\lambda_{\max}$) denotes the smallest (respectively the largest) eigenvalue of the positive semidefinite matrix $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$. Then,

(a) The matrix $\mathbf{G}(\zeta)$ has essentially bounded entries on $\mathbb{T}$ if and only if $B_{\mathbf{G}}<\infty$.

(b) There exist constants $0<A\le B<\infty$ such that $A\,\mathbb{I}_r\le \mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\le B\,\mathbb{I}_r$, a.e. $\zeta\in\mathbb{T}$ if and only if $0<A_{\mathbf{G}}\le B_{\mathbf{G}}<\infty$.

Proof. The first part of lemma follows from that $\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]=\|\mathbf{G}(\zeta)\|_2^2$, and

$$\max_{i,j}|a_{ij}|\le\|\mathbf{A}\|_2\le\sqrt{mn}\ \ \max_{i,j}|a_{ij}| \qquad \text{for any matrix } \mathbf{A}=\big[a_{ij}\big]_{\substack{i=1,2\dots,m\\ j=1,2\dots,n}}\,,$$

where $\|\mathbf{A}\|_2$ denotes the spectral norm of the matrix $\mathbf{A}$ (see, for instance, [13])

Now we prove the second part of the lemma. Since $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\le B\,\mathbb{I}_r$ means that $\langle Bx-\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x,x\rangle\ge 0$ for all $x\in\mathbb{C}^r$, in particular, taking an eigenvector x associated to the largest eigenvalue $\lambda_{\max}$ of $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$ such that $\|x\|=1$, one has that $B\ge\lambda_{\max}(\mathbf{G}^*(\zeta)\mathbf{G}(\zeta))$. Hence, $B\ge\operatorname{ess\,sup}_{\zeta\in\mathbb{T}}\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]$. In a similar way, $A\,\mathbb{I}_r\le\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$ implies that $A\le\operatorname{ess\,inf}_{\zeta\in\mathbb{T}}\lambda_{\min}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]$.
Conversely, Rayleigh-Ritz theorem [13, p. 176] yields that

$$\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]=\max_{x\in\mathbb{C}^r}\frac{x^*\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)}{x^*x}=\max_{x\in\mathbb{C}^r}\frac{\langle\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x,x\rangle}{\langle x,x\rangle}$$

Thus, $\operatorname{ess\,sup}_{\zeta\in\mathbb{T}}\lambda_{\max}\big[\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\big]=B_{\mathbf{G}}$ implies that

$$\max_{x\in\mathbb{C}^r}\frac{\langle\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)x,x\rangle}{\langle x,x\rangle}\le B_{\mathbf{G}}\,,\qquad \textit{a.e. } \zeta\in\mathbb{T}\,.$$

In other words, $B_{\mathbf{G}}\mathbb{I}_r\ge\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)$; analogously, $\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\ge A_{\mathbf{G}}\mathbb{I}_r$.
It is easy to deduce from the proof that $A_{\mathbf{G}}$ and $B_{\mathbf{G}}$ are the optimal constants $A>0$ and $B<\infty$ satisfying the inequalities $A\,\mathbb{I}_r\le\mathbf{G}^*(\zeta)\mathbf{G}(\zeta)\le B\,\mathbb{I}_r$, a.e. $\zeta\in\mathbb{T}$. □

As a consequence of Lemma 6, statements (b) and (c) in Theorem 5 can be restated in terms of the constants

$$A_{\mathbf{\Psi}} := \operatorname*{ess\,inf}_{\zeta\in\mathbb{T}} \ \lambda_{\min}\big[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\big];\ \ B_{\mathbf{\Psi}} := \operatorname*{ess\,sup}_{\zeta\in\mathbb{T}} \ \lambda_{\max}\big[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\zeta)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)\big] \tag{20}$$

as:

Theorem 7. *Let $b_j\in\mathcal{A}_a$ for $j=1,2,\dots,s$, and let $\Psi_{\mathbf{a},\mathbf{b}}$ be the associated matrix given in* (9) *and its related constants* (20)*. Then, the following results hold:*

(i) The sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a Bessel sequence for $\mathcal{A}_a$ if and only if the constant $B_{\mathbf{\Psi}}<\infty$.

(ii) The sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$ if and only if the constants $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ satisfy $0<A_{\mathbf{\Psi}}\le B_{\mathbf{\Psi}}<\infty$. In this case, $A_{\mathbf{\Psi}}$ and $B_{\mathbf{\Psi}}$ are the optimal frame bounds for $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\,j=1,2,\dots s}$.

2.3 The frame expansion

Define the $r \times s$ matrix of functions on the torus $\mathbb{T}$

$$\mathbf{\Gamma}(\mathrm{e}^{\mathrm{i}\theta}) := \sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \, \mathrm{e}^{\mathrm{i}k\theta} = \big[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\big]^{-1}\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) \tag{21}$$

The following expansion involving the inner products $\alpha_{n,j} = \langle x, U^{rn} b_j\rangle$ of $x \in \mathcal{A}_a$ holds:

Lemma 8. *Assume that the matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\zeta)$ has essentially bounded entries on $\mathbb{T}$. For any $x = \sum_{k\in\mathbb{Z}} x_k U^k a \in \mathcal{A}_a$ we have*

$$\widetilde{\mathbf{x}}_{\mathbf{n}} = \sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \, \boldsymbol{\alpha}_{n-k} \,,$$

where $\widetilde{\mathbf{x}}_{\mathbf{n}}$ denotes the n-th Fourier coefficient of the function $\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})$ defined in (14), and the sequence $\{\boldsymbol{\alpha}_n\}_{n\in\mathbb{Z}}$ is given in (10).

Proof. Indeed,

$$\begin{aligned}
\widetilde{\mathbf{x}}_{\mathbf{n}} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta} d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \big(\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \mathrm{e}^{\mathrm{i}k\theta}\big)\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{-\mathrm{i}n\theta} d\theta \\
&= \sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\widetilde{\mathbf{X}}(\mathrm{e}^{i\theta})\mathrm{e}^{-\mathrm{i}(n-k)\theta} d\theta = \sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \, \boldsymbol{\alpha}_{n-k} \,.
\end{aligned}$$

□

At this point, we are ready to prove the following expansion result:

Theorem 9. *Let $b_j \in \mathcal{A}_a$ for $j = 1, 2, \ldots, s$, and assume that the associated matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}$ given in (9) has essentially bounded entries on $\mathbb{T}$, i.e., $B_{\mathbf{\Psi}} < \infty$. The following statements are equivalent:*

(i) The constant $A_{\mathbf{\Psi}} > 0$.

(ii) There exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, such that the sequence $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$, yielding, for any $x \in \mathcal{A}_a$, the expansion

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \langle x, U^{rk} b_j\rangle \, U^{rk} c_j \quad \text{in } \mathcal{H}\,. \tag{22}$$

In case the equivalent conditions hold, $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ and $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ form a pair of dual frames in $\mathcal{A}_a$.

Proof. First we prove that (i) implies (ii). Observe that $x = \sum_{k\in\mathbb{Z}} x_k \, U^k a$ can be written as $\sum_{n\in\mathbb{Z}} \widetilde{\mathbf{x}}_{\mathbf{n}}^\top \, \widetilde{\mathbf{a}}_{\mathbf{n}}$ where $\widetilde{\mathbf{a}}_{\mathbf{n}} = \big(U^{nr} a, U^{nr+1} a, \cdots, U^{nr+r-1} a\big)^\top$. Next,

$$\begin{aligned}
x &= \sum_{n\in\mathbb{Z}} \widetilde{\mathbf{x}}_{\mathbf{n}}^\top \widetilde{\mathbf{a}}_{\mathbf{n}} = \sum_{n\in\mathbb{Z}} \big(\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k \boldsymbol{\alpha}_{n-k}\big)^\top \widetilde{\mathbf{a}}_{\mathbf{n}} = \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_{n-k}^\top \mathbf{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n}} \\
&= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \mathbf{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n}+\mathbf{k}} = \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \big(\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n}+\mathbf{k}}\big)
\end{aligned} \tag{23}$$

For $l \in \mathbb{Z}$ and $j = 1, 2, \ldots, s$ define $c_{l,j} := U^{rl} c_j$, where $\big(c_1, c_2, \ldots, c_s\big)^\top = \sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k^\top \, \widetilde{\mathbf{a}}_{\mathbf{k}}$, and $b_{l,j} := U^{rl} b_j$. Then Eq. (23) implies

$$\begin{aligned} x &= \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top \big(\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{n+k}} \big) = \sum_{n\in\mathbb{Z}} \boldsymbol{\alpha}_n^\top U^{nr} \Big(\sum_{k\in\mathbb{Z}} \mathbf{\Gamma}_k^\top \widetilde{\mathbf{a}}_{\mathbf{k}} \Big) \\ &= \sum_{l=1}^{s} \sum_{n\in\mathbb{Z}} \langle x, b_{n,l} \rangle \, c_{n,l} \quad \text{in } \mathcal{H} \,. \end{aligned} \tag{24}$$

In order to be allowed to use [7, Lemma 5.6.2], we have to prove that the above constructed sequence $\{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$ is a Bessel sequence for $\mathcal{A}_a$. To this end, we compute the corresponding $\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}$ matrix for $\mathbf{c} := \{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots,s}$. Setting $[\mathbf{\Gamma}_k^\top]_{ij} = a_{ij}^k$, we obtain

$$\begin{aligned} \langle U^k a, c_{n,j} \rangle &= \sum_{l\in\mathbb{Z}} \sum_{i=1}^{r} \langle U^k a, U^{nr} \big(a_{ji}^l \, U^{lr+ir+i-1} a \big) \rangle = \sum_{l\in\mathbb{Z}} \sum_{i=1}^{r} \overline{a}_{ji}^l \, \langle U^{k-nr-lr-i+1} a, a \rangle \\ &= \sum_{l\in\mathbb{Z}} \sum_{i=1}^{r} \overline{a}_{ji}^l \, \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr-lr-i+1)\theta} \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \,. \end{aligned}$$

Now,

$$\begin{aligned} \begin{pmatrix} \langle U^k a, c_{n,1} \rangle \\ \langle U^k a, c_{n,2} \rangle \\ \vdots \\ \langle U^k a, c_{n,s} \rangle \end{pmatrix} &= \sum_{l\in\mathbb{Z}} \overline{\mathbf{\Gamma}}_l^\top \begin{pmatrix} \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr-lr)\theta} \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr-lr-1)\theta} \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \\ \vdots \\ \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr-lr-r+1)\theta} \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \end{pmatrix} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} \overline{\mathbf{\Gamma}}_l^\top \mathrm{e}^{-\mathrm{i}lr\theta} \begin{pmatrix} \mathrm{e}^{\mathrm{i}(k-nr)\theta} \\ \mathrm{e}^{\mathrm{i}(k-nr-1)\theta} \\ \vdots \\ \mathrm{e}^{\mathrm{i}(k-nr-r+1)\theta} \end{pmatrix} \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}(k-nr)\theta} \mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) \widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta}) \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) d\theta \,, \end{aligned}$$

where $\widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta}) := \big(1, \mathrm{e}^{-\mathrm{i}\theta}, \ldots, \mathrm{e}^{-\mathrm{i}(r-1)\theta}\big)^\top$. Hence, we have deduced that

$$\mathbf{\Phi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) = \mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) \widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta}) \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) \,.$$

Therefore, for $l = 0, 1, \ldots, r-1$, we have

$$\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}^l(\mathrm{e}^{\mathrm{i}\theta}) := D_r S^{-l} \big[\mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) \widetilde{\mathbf{e}}(\mathrm{e}^{\mathrm{i}\theta}) \phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) \big] \,,$$

and consequently, the $s \times r$ matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) := \Big(\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}^0(\mathrm{e}^{\mathrm{i}\theta}), \mathbf{\Psi}_{\mathbf{a},\mathbf{c}}^1(\mathrm{e}^{\mathrm{i}\theta}), \ldots, \mathbf{\Psi}_{\mathbf{a},\mathbf{c}}^{r-1}(\mathrm{e}^{\mathrm{i}\theta}) \Big)$ can be written as

$$\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta}) = D_r \big[\phi_{\mathbf{a}}(\mathrm{e}^{\mathrm{i}\theta}) \mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) \widetilde{\mathbf{E}}(\mathrm{e}^{\mathrm{i}\theta}) \big] \,, \tag{25}$$

where

$$\widetilde{\mathbf{E}}(\mathrm{e}^{\mathrm{i}\theta}) := \begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\theta} & \cdots & \mathrm{e}^{\mathrm{i}(r-1)\theta} \\ \mathrm{e}^{-\mathrm{i}\theta} & 1 & \cdots & \mathrm{e}^{\mathrm{i}(r-2)\theta} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{e}^{-\mathrm{i}(r-1)\theta} & \mathrm{e}^{-\mathrm{i}(r-2)\theta} & \cdots & 1 \end{pmatrix}.$$

As a consequence of Theorem 7, the proof ends if we prove that the matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta})$ has essentially bounded entries: Clearly, the decimation operator D_r sends bounded functions into bounded functions; Theorem 3 implies that $\phi_{\mathbf{a}}$ is bounded so, taking into account (25) it remains to check that the matrix $\mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})$ has essentially bounded entries.
Now, $\mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta}) = \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big]^{-1}$, the lower bound condition (c) in Theorem 5 and Lemma 6 imply that $\big[\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}r\theta})\big]^{-1}$ has bounded entries, and therefore the matrix $\mathbf{\Gamma}^*(\mathrm{e}^{\mathrm{i}r\theta})$ has bounded entries. We have shown that $\mathbf{\Psi}_{\mathbf{a},\mathbf{c}}(\mathrm{e}^{\mathrm{i}\theta})$ has bounded entries, then Theorem 7, part (a) and Lemma 6 guarantee that the sequence $\{c_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Bessel sequence; then, the sequences $\{b_{k,j}\}_{k\in\mathbb{Z},\, j=1,2,\dots,s}$ and $\{c_{k,j}\}_{k\in\mathbb{Z},\, j=1,2,\dots,s}$ form a pair of dual frames in $\mathcal{A}_a$ (see [7, Lemma 5.6.2]).

Finally, condition (ii) implies condition (i). According to [7, Lemma 5.6.2], the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$ since it is a Bessel sequence and the expansion in (ii) holds. By using Theorem 7 we obtain that $A_{\mathbf{\Psi}} > 0$. □

It is worth to observe that the analysis done in Theorem 9 provides a whole family of dual frames for the sequence $\big\{U^{rk}b_j\big\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$. In fact, everything works if we replace $\mathbf{\Gamma}(\mathrm{e}^{\mathrm{i}\theta})$ in (21) by any matrix of the form,

$$\mathbf{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta}) := \mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) + \mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})\big[\mathbb{I}_s - \mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\big]\,, \tag{26}$$

where $\mathbb{U}(\mathrm{e}^{\mathrm{i}\theta})$ is any $r\times s$ matrix with entries in $L^\infty(\mathbb{T})$, and $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}$ denotes the Moore-Penrose pseudo-inverse $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta}) := [\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})]^{-1}\mathbf{\Psi}^*_{\mathbf{a},\mathbf{b}}(\mathrm{e}^{\mathrm{i}\theta})$. Note that we need essentially bounded entries in the matrix $\mathbf{\Gamma}_{\mathbb{U}}(\mathrm{e}^{\mathrm{i}\theta})$ since the multiplication operator $M_F : f \mapsto Ff$ in $L^2(\mathbb{T})$ is well-defined (and consequently bounded) if and only if $F \in L^\infty(\mathbb{T})$.

Notice that if $s = r$, we have $\mathbf{\Psi}^{\dagger}_{\mathbf{a},\mathbf{b}} = \mathbf{\Psi}^{-1}_{\mathbf{a},\mathbf{b}}$ which implies a unique $\mathbf{\Gamma}_{\mathbb{U}}$, and we are in presence of a pair of dual Riesz bases. In fact, the following result holds:

Corollary 10. *Let $b_j \in \mathcal{A}_a$ for $j = 1,2,\dots,r$, i.e., $r = s$ in Theorem 9. Assume that the square matrix $\mathbf{\Psi}_{\mathbf{a},\mathbf{b}}$ given in* (9) *has entries essentially bounded on $\mathbb{T}$, i.e., $B_{\mathbf{\Psi}} < \infty$. The following statements are equivalent:*

(a) The constant $A_{\mathbf{\Psi}} > 0$.

(b) There exists a Riesz basis $\{C_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ such that for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}} \langle x, U^{rk}b_j\rangle\, C_{k,j} \quad \text{in } \mathcal{H}$$

holds.

In case the equivalent conditions are satisfied, necessarily there exist $c_j \in \mathcal{A}_a$*,* $j = 1, 2, \ldots, r$*, such that* $C_{k,j} = U^{rk} c_j$ *for* $k \in \mathbb{Z}$ *and* $j = 1, 2, \ldots, r$*. Moreover, the sequences* $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ *and* $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ *are dual Riesz bases in* $\mathcal{A}_a$*, and the interpolation property* $\langle c_j, U^{rk} b_{j'}\rangle = \delta_{j,j'}\,\delta_{k,0}$*, where* $k \in \mathbb{Z}$ *and* $j, j' = 1, 2, \ldots, r$*, holds.*

Proof. To prove $(a) \Rightarrow (b)$ we use Theorem 9; whenever $0 < A_{\mathbf{\Psi}} \le B_{\mathbf{\Psi}} < \infty$ there exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \ldots, s$, such that the sequence $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a frame for $\mathcal{A}_a$ and, for any $x \in \mathcal{A}_a$ the expansion

$$x = \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} \langle x, U^{rk} b_j\rangle\, U^{rk} c_j \quad \text{in } \mathcal{H}\,,$$

holds. Actually, from Theorem 5 we get that $r = s$ implies that $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis, and consequently, $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is indeed its dual Riesz basis.
The converse follows easily from the fact that if $\{C_{k,j}\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is a Riesz basis, then (b) implies that $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\ldots s}$ is its dual Riesz basis; hence, Theorem 5 provides $A_{\mathbf{\Psi}} > 0$. The interpolation property comes out from the biorthogonal condition of a pair of dual Riesz bases. □

Closing this section it is worth to mention that the results stated and proved in Sections 2.2 and 2.3 mathematically enrich some of the remarkable results concerning regular sampling contained in the interesting Ref. [21]. Here we have assumed only one generator $a \in \mathcal{H}$ and that $b_j \in \mathcal{A}_a$ for all $j = 1, 2, \ldots, s$. If $b_j \notin \mathcal{A}_a$ for some j, see the additional remarks in Section 4. The case of several generators $a_l \in \mathcal{H}$, $l = 1, 2, \ldots, L$, can be essentially treated in the same way.

3 Some perturbation results

In Section 2 we have only used the discrete group of unitary operators $\{U^n\}_{n\in\mathbb{Z}}$ which is completely determined by the unitary operator U. In order to carry out the study of some perturbation results stated below we need the availability of a continuous group of unitary operators $\{U^t\}_{t\in\mathbb{R}}$ which includes the unitary operator U, say, for instance, that $U := U^1$.

3.1 On continuous groups of unitary operators

Let $\{U^t\}_{t\in\mathbb{R}}$ denote a continuous *group of unitary operators* in $\mathcal{H}$ such that $U := U^1$. Recall that $\{U^t\}_{t\in\mathbb{R}}$ is a family of unitary operators in $\mathcal{H}$ satisfying (see Ref. [2, vol. 2; p. 29]):

(1) $U^t\, U^{t'} = U^{t+t'}$,

(2) $U^0 = I_{\mathcal{H}}$,

(3) $\langle U^t x, y\rangle_{\mathcal{H}}$ is a continuous function of t for any $x, y \in \mathcal{H}$.

Note that $(U^t)^{-1} = U^{-t}$, and since $(U^t)^* = (U^t)^{-1}$, we have $(U^t)^* = U^{-t}$.

Classical Stone's theorem [23] assures us the existence of a self-adjoint operator T (possibly unbounded) such that $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$. This self-adjoint operator T, defined on the dense domain of $\mathcal{H}$

$$D_T := \left\{ x \in \mathcal{H} \ \text{ such that } \int_{-\infty}^{\infty} w^2 \, d\|E_w x\|^2 < \infty \right\},$$

admits the *spectral representation* $T = \int_{-\infty}^{\infty} w \, dE_w$ which means

$$\langle Tx, y \rangle = \int_{-\infty}^{\infty} w \, d\langle E_w x, y \rangle \qquad \text{for any } x \in D_T \text{ and } y \in \mathcal{H}\,,$$

where $\{E_w\}_{w \in \mathbb{R}}$ is the corresponding *resolution of the identity*, i.e., a one-parameter family of projection operators E_w in $\mathcal{H}$ such that

(i) $E_{-\infty} := \lim_{w \to -\infty} E_w = O_{\mathcal{H}}, \quad E_{\infty} := \lim_{w \to \infty} E_w = I_{\mathcal{H}}$,

(ii) $E_{w^-} = E_w$ for every $-\infty < w < \infty$,

(iii) $E_u \, E_v = E_w$ where $w = \min\{u, v\}$.

Recall that $\|E_w x\|^2$ and $\langle E_w x, y \rangle$, as functions of w, have bounded variation and define, respectively, a positive and a complex Borel measure on $\mathbb{R}$.

Furthermore, for any $x \in D_T$ we have that $\lim_{t \to 0} \dfrac{U^t x - x}{t} = \mathrm{i}\, Tx$ and the operator $\mathrm{i}\, T$ is said to be the *infinitesimal generator* of the group $\{U^t\}_{t \in \mathbb{R}}$. For each $x \in D_T$, $U^t x$ is a continuous differentiable function of t. Notice that, whenever the self-adjoint operator T is bounded, $D_T = \mathcal{H}$ and $\mathrm{e}^{\mathrm{i}tT}$ can be defined as the usual exponential series; in any case, $U^t \equiv \mathrm{e}^{\mathrm{i}tT}$ means that

$$\langle U^t x, y \rangle = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}wt} d\langle E_w x, y \rangle\,, \quad t \in \mathbb{R}\,,$$

where $x \in D_T$ and $y \in \mathcal{H}$. A comment on the continuity of a group of unitary operators is in order: The group is said to be *strongly continuous* if, for each $x \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, $U^t x \to U^{t_0} x$ as $t \to t_0$. If $\mathcal{H}$ is a separable Hilbert space, strong continuity can be deduced from continuity and even from weak measurability, i.e., $\langle U^t x, y \rangle_{\mathcal{H}}$ is a Lebesgue measurable function of t for any $x, y \in \mathcal{H}$.
The following result taken from [2, vol.2; p. 24] will be used later: For $x \in D_T$ and $y \in \mathcal{H}$, the inequality

$$\Big| \int_{-\infty}^{\infty} \varphi(w) d\langle E_w x, y \rangle \Big| \leq \|y\| \sqrt{\int_{-\infty}^{\infty} |\varphi(w)|^2 d\langle E_w x, x \rangle}, \tag{27}$$

holds, where φ is a real or complex function which is continuous in $\mathbb{R}$ with the possible exception of a finite number of points.
For the details on the theory of continuous groups of unitary operators, see Refs. [2, 6, 29, 30].

3.2 Studying the perturbed sequence $\{U^{rk+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$

Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$, consider the corresponding perturbed sequence $\{U^{rk+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z}\,;j=1,2,\dots s}$. Consider the following perturbation result (see [7, p. 354] for the proof):

Lemma 11. *Let $\{x_n\}_{n=1}^\infty$ be a frame for the Hilbert space $\mathcal{H}$ with frame bounds A, B, and let $\{y_n\}_{n=1}^\infty$ be a sequence in $\mathcal{H}$. If there exists a constant $R<A$ such that*

$$\sum_{n=1}^{\infty} \big|\langle x_n - y_n, x\rangle\big|^2 \le R\|x\|^2 \ \text{ for each } x\in\mathcal{H}\,,$$

then the sequence $\{y_n\}_{n=1}^\infty$ is also a frame for $\mathcal{H}$ with bounds $A\big(1-\sqrt{R/A}\big)^2$ and $B\big(1+\sqrt{R/B}\big)^2$. If $\{x_n\}_{n=1}^\infty$ is a Riesz basis, then $\{y_n\}_{n=1}^\infty$ is a Riesz basis.

Note that it cannot be directly applied to the sequences $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ and $\{U^{rk+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ since the first one is not a frame for the entire Hilbert space $\mathcal{H}$, and its perturbed sequence does not necessarily belong to the subspace $\mathcal{A}_a$. However, something can be said in case $\{U^{rk}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots r}$ is a Riesz sequence in $\mathcal{H}$:

Theorem 12. *Assume that for some $b_j\in D_T$, i.e., $\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2<\infty$ for each $1\le j\le r$, the sequence $\{U^{kr}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz basis for $\mathcal{A}_a$ with Riesz bounds $0<A_{\boldsymbol{\Psi}}\le B_{\boldsymbol{\Psi}}<\infty$. For a sequence $\boldsymbol{\epsilon} := \{\epsilon_{kj}\}_{k\in\mathbb{Z},\, j=1,2,\dots,r}$ of errors, let R be the constant given by*

$$R := \|\boldsymbol{\epsilon}\|^2 \max_{j=1,2,\dots,r} \Big\{ \int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2 \Big\}\,,$$

where $\|\boldsymbol{\epsilon}\|$ denotes the ℓ^2_r-norm of the sequence $\boldsymbol{\epsilon}$.
If $R<A_{\boldsymbol{\Psi}}$, then the sequence $\{U^{kr+\epsilon_{kj}}b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots,r}$ is a Riesz sequence in $\mathcal{H}$ with Riesz bounds $A_{\boldsymbol{\Psi}}\big(1-\sqrt{R/A_{\boldsymbol{\Psi}}}\big)^2$ and $B_{\boldsymbol{\Psi}}\big(1+\sqrt{R/B_{\boldsymbol{\Psi}}}\big)^2$.

Proof. By using (27) we have

$$\begin{aligned}
\Big|\langle x, U^{kr}b_j - U^{kr+\epsilon_{kj}}b_j\rangle\Big| &= \Big|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw} d\langle E_w x, b_j\rangle - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw - i\epsilon_{kj}w} d\langle E_w x, b_j\rangle\Big| \\
&= \Big|\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}krw}(1-\mathrm{e}^{-\mathrm{i}\epsilon_{kj}w}) d\langle E_w x, b_j\rangle\Big| = \Big|\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}krw}(1-\mathrm{e}^{\mathrm{i}\epsilon_{kj}w}) d\langle E_w b_j, x\rangle\Big| \\
&\le \|x\| \sqrt{\int_{-\infty}^{\infty} |1-\mathrm{e}^{\mathrm{i}\epsilon_{kj}w}|^2 d\|E_w b_j\|^2} \le \|x\| \sqrt{\int_{-\infty}^{\infty} w^2 |\epsilon_{kj}|^2 d\|E_w b_j\|^2} \\
&= |\epsilon_{kj}|\|x\| \sqrt{\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2}\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} |\langle x, U^{kr}b_j - U^{kr+\epsilon_{kj}}b_j\rangle|^2 &\le \|x\|^2 \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} \Big(\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2\Big)|\epsilon_{kj}|^2 \\
&\le \|x\|^2 \max_{j=1,2,\dots,r}\Big\{\int_{-\infty}^{\infty} w^2 d\|E_w b_j\|^2\Big\} \sum_{j=1}^{r}\sum_{k\in\mathbb{Z}} |\epsilon_{kj}|^2
\end{aligned}$$

Hence, Lemma 11 and Theorem 15.3.2 in [7, p. 356] give the desired result. □

3.3 On the perturbed frame expansion

Next, we deal with the problem of the recovery of any $x \in \mathcal{A}_a$ in a stable way from the perturbed sequence

$$\big\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,,$$

where $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ denotes a sequence of real errors. In order to face this problem, we propose a possible strategy: Let $\mathcal{T}_{U,a} : L^2(0,1) \longrightarrow \mathcal{A}_a$ be the isomorphism which maps the orthonormal basis $\big\{\mathrm{e}^{2\pi i n w}\big\}_{n\in\mathbb{Z}}$ onto the Riesz basis $\{U^n a\}_{n\in\mathbb{Z}}$ for $\mathcal{A}_a$. In other words:

$$\begin{array}{rccl} \mathcal{T}_{U,a}: & L^2(0,1) & \longrightarrow & \mathcal{A}_a \\ & F=\displaystyle\sum_{n\in\mathbb{Z}} \alpha_n\, \mathrm{e}^{2\pi i n w} & \longmapsto & x=\displaystyle\sum_{n\in\mathbb{Z}} \alpha_n\, U^n a\,. \end{array}$$

Thus,

$$\begin{aligned} \langle x, U^t b_j\rangle_{\mathcal{H}} &= \Big\langle \sum_{n\in\mathbb{Z}} \alpha_n\, U^n a, U^t b_j\Big\rangle_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} \alpha_n\, \overline{\langle U^t b_j, U^n a\rangle}_{\mathcal{H}} \\ &= \Big\langle F, \sum_{n\in\mathbb{Z}} \langle U^t b_j, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w}\Big\rangle_{L^2(0,1)} = \langle F, K_j^t\rangle_{L^2(0,1)}\,, \end{aligned} \tag{28}$$

where $\mathcal{T}_{U,a}F = x$, and the function $K_j^t(w) := \sum_{n\in\mathbb{Z}} \langle U^t b_j, U^n a\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i n w}$ belongs to $L^2(0,1)$ since the sequence $\big\{\langle U^t b, U^n a\rangle_{\mathcal{H}}\big\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$ for each $t \in \mathbb{R}$.

Hence, for any $x \in \mathcal{A}_a$ we have the expressions:

$$\langle x, U^{rm} b_j\rangle_{\mathcal{H}} = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\rangle_{L^2(0,1)}\,;\ \langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}} = \big\langle F, \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi i r m w}\big\rangle_{L^2(0,1)}\,,$$

where the functions

$$g_j(w) := \sum_{k\in\mathbb{Z}} \langle a, U^k b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i k w} \quad \text{and} \quad g_{m,j}(w) := \sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi i k w} \tag{29}$$

belong to $L^2(0,1)$. Therefore, we can see the sequence $\big\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ as a perturbation of the sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$. From [11, Lemma 3] we know that this sequence is a frame for $L^2(0,1)$ if and only if $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$ where the constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ are given by

$$\alpha_{\mathbb{G}} := \operatorname*{ess\,inf}_{w\in(0,1/r)} \lambda_{\min}[\mathbb{G}^*(w)\mathbb{G}(w)], \quad \beta_{\mathbb{G}} := \operatorname*{ess\,sup}_{w\in(0,1/r)} \lambda_{\max}[\mathbb{G}^*(w)\mathbb{G}(w)]\,, \tag{30}$$

and $\mathbb{G}(w)$ is the $s \times r$ matrix

$$\mathbb{G}(w) := \Big[g_j\Big(w + \frac{k-1}{r}\Big)\Big]_{\substack{j=1,2,\dots,s\\ k=1,2,\dots,r}}\,.$$

Besides, the optimal frame bounds for $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$.

Given an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ we define

$$d_{m,k}^{(j)} := \langle a, U^{rm-k+\epsilon_{mj}} b_j\rangle - \langle a, U^{rm-k} b_j\rangle\,,$$

For any sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ we have

$$\begin{aligned}\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}} d_{m,k}^{(j)}\; c_k\Big|^2 &\le \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\sum_{l,k\in\mathbb{Z}} |d_{m,l}^{(j)} c_l \overline{d}_{m,k}^{(j)} \overline{c}_k| \\ = \sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}} |c_l|\;|c_k| \sum_{m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}| &\le \sum_{j=1}^{s}\sum_{l,k\in\mathbb{Z}} \frac{|c_l|^2+|c_k|^2}{2} \sum_{m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}| \qquad (31)\\ = \sum_{j=1}^{s}\sum_{l\in\mathbb{Z}} |c_l|^2 \sum_{k,m\in\mathbb{Z}} |d_{m,l}^{(j)} d_{m,k}^{(j)}|\,. \end{aligned}$$

Now, for $|\gamma| < 1/2$ define the functions,

$$M_{a,b_j}(\gamma) := \sum_{k\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{k+t} b_j\rangle - \langle a, U^k b_j\rangle|\,,$$

and

$$N_{a,b_j}(\gamma) := \max_{k=0,1,\dots,r-1} \sum_{m\in\mathbb{Z}} \max_{t\in[-\gamma,\gamma]} |\langle a, U^{rm+k+t} b_j\rangle - \langle a, U^{rm+k} b_j\rangle|\,.$$

Notice that $N_{a,b_j}(\gamma) \le M_{a,b_j}(\gamma)$ and for $r = 1$ the equality holds. Moreover, assuming that the continuous functions $\varphi_j(t) := \langle a, U^t b_j\rangle$, $j = 1,2,\dots,s$, satisfy a decay condition as $\varphi_j(t) = O\big(|t|^{-(1+\eta_j)}\big)$ when $|t|\to\infty$ for some $\eta_j > 0$, we may deduce that the functions $N_{a,b_j}(\gamma)$ and $M_{a,b_j}(\gamma)$ are continuous near to 0.

Theorem 13. *Assume that for the functions g_j, $j = 1,2,\dots,s$, given in* (29) *we have $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$. For an error sequence $\boldsymbol{\epsilon} := \{\epsilon_{mj}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$, define the constant $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j = 1,2,\dots,s$. Then the condition*

$$\sum_{j=1}^{s} M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$$

implies that there exists a frame $\{C_{m,j}^{\boldsymbol{\epsilon}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ for $\mathcal{A}_a$ such that, for any $x\in\mathcal{A}_a$, the sampling expansion

$$x = \sum_{j=1}^{s}\sum_{m\in\mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, C_{m,j}^{\boldsymbol{\epsilon}} \quad \text{in } \mathcal{H}\,, \qquad (32)$$

holds. Moreover, when $r = s$ the sequence $\{C_{m,j}^{\boldsymbol{\epsilon}}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a Riesz basis for $\mathcal{A}_a$, and the interpolation property $\big\langle C_{n,l}^{\boldsymbol{\epsilon}}, U^{rm+\epsilon_{mj}} b_j\big\rangle_{\mathcal{H}} = \delta_{l,j}\,\delta_{n,m}$ holds.

Proof. The sequence $\big\{\overline{g_j(w)}\, \mathrm{e}^{2\pi i rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame (a Riesz basis if $r = s$) for $L^2(0,1)$ with frame (Riesz) bounds $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$. For any $F(w) = \sum_{l\in\mathbb{Z}} a_l\, \mathrm{e}^{2\pi i lw}$ in $L^2(0,1)$

we have

$$
\begin{aligned}
&\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rm\cdot}-\overline{g_j(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rm\cdot},F(\cdot)\big\rangle_{L^2(0,1)}\big|^2\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \sum_{k\in\mathbb{Z}}\big(\overline{\langle a,\,U^{k+\epsilon_{mj}}b_j\rangle}-\overline{\langle a,\,U^{k}b_j\rangle}\big)\mathrm{e}^{2\pi\mathrm{i}(rm-k)\cdot},F(\cdot)\big\rangle_{L^2(0,1)}\big|^2\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \sum_{k\in\mathbb{Z}}\big(\overline{\langle a,\,U^{rm-k+\epsilon_{mj}}b_j\rangle}-\overline{\langle a,\,U^{rm-k}b_j\rangle}\big)\mathrm{e}^{2\pi\mathrm{i}k\cdot},F(\cdot)\big\rangle_{L^2(0,1)}\big|^2 \qquad (33)\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\sum_{k\in\mathbb{Z}}\big(\overline{\langle a,\,U^{rm-k+\epsilon_{mj}}b_j\rangle}-\overline{\langle a,\,U^{rm-k}b_j\rangle}\big)\,\overline{a}_k\big|^2\\
&=\sum_{j=1}^{s}\sum_{m\in\mathbb{Z}}\Big|\sum_{k\in\mathbb{Z}}d_{m,k}^{(j)}\;a_k\Big|^2 .
\end{aligned}
$$

From (31) and the definition of the functions M_{a,b_j} and N_{a,b_j} we obtain

$$
\begin{aligned}
\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big|\big\langle \overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rm\cdot}-\overline{g_j(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rm\cdot},F(\cdot)\big\rangle_{L^2(0,1)}\big|^2 &\le \sum_{j=1}^{s}M_{a,b_j}(\gamma_j)N_{a,b_j}(\gamma_j)\|\{a_l\}_{l\in\mathbb{Z}}\|^2\\
&\le \sum_{j=1}^{s}M_{a,b_j}(\gamma_j)N_{a,b_j}(\gamma_j)\|F\|^2_{L^2(0,1)}
\end{aligned}
$$

By using Lemma 11 we obtain that the sequence $\big\{\overline{g_{m,j}(w)}\,\mathrm{e}^{2\pi\mathrm{i}rmw}\big\}_{m\in\mathbb{Z};\,j=1,2,\dots s}$ is a frame for $L^2(0,1)$ (a Riesz basis if $r=s$). Let $\{h^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ be its canonical dual frame. Hence, for any $F\in L^2(0,1)$

$$
\begin{aligned}
F&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\big\langle F(\cdot),\overline{g_{m,j}(\cdot)}\,\mathrm{e}^{2\pi\mathrm{i}rm\cdot}\big\rangle_{L^2(0,1)}\,h^{\boldsymbol{\epsilon}}_{m,j}\\
&=\sum_{m\in\mathbb{Z}}\sum_{j=1}^{s}\langle x,U^{rm+\epsilon_{mj}}b_j\rangle_{\mathcal{H}}\,h^{\boldsymbol{\epsilon}}_{m,j}\,.
\end{aligned}
$$

Applying the isomorphism $\mathcal{T}_{U,a}$, one gets (32), where $C^{\boldsymbol{\epsilon}}_{m,j}=\mathcal{T}_{U,a}\big(h^{\boldsymbol{\epsilon}}_{m,j}\big)$. Since $\mathcal{T}_{U,a}$ is an isomorphism between $L^2(0,1)$ and $\mathcal{A}_a$, the sequence $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ is a frame for $\mathcal{A}_a$ (a Riesz basis if $r=s$). The interpolatory property in the case $r=s$ follows from the uniqueness of the coefficients with respect to a Riesz basis. □

3.4 A frame algorithm in $\ell^2(\mathbb{Z})$

Sampling formula (32) is useless from a practical point of view: it is impossible to determine the involved frame $\{C^{\boldsymbol{\epsilon}}_{m,j}\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$. As a consequence, in order to recover $x\in\mathcal{A}_a$ from the sequence of inner products $\big\{\langle x,U^{rm+\epsilon_{mj}}b_j\rangle_{\mathcal{H}}\big\}_{m\in\mathbb{Z};\,j=1,2,\dots,s}$ we should implement a frame algorithm in $\ell^2(\mathbb{Z})$. Another possibility is given in the recent Ref. [1].

Next we are going to implement a frame algorithm in the $\ell^2(\mathbb{Z})$ setting. To this end, consider the canonical isometry

$$\widetilde{\mathcal{U}} : \ell^2(\mathbb{Z}) \longrightarrow L^2(0,1) \quad \text{such that} \quad \widetilde{\mathcal{U}}\, c := \sum_{k\in\mathbb{Z}} c_k\, \mathrm{e}^{2\pi \mathrm{i} k w} \text{ for } c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\,.$$

For $x = \sum_{n\in\mathbb{Z}} c_n U^n a \in \mathcal{A}_a$, denote by $\mathbb{F}$ the sequence

$$\mathbb{F} := \widetilde{\mathcal{U}}^{-1}\, \mathcal{T}_{U,a}^{-1}\, x = \{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\,.$$

The inner products $\big\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ can be written as

$$\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}} = \big\langle \mathcal{T}_{U,a}^{-1}\, x,\ \overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\rangle_{L^2(0,1)} = \big\langle \mathbb{F}, \mathbb{L}_{m,j}\big\rangle_{\ell^2(\mathbb{Z})}$$

where, for $j = 1, 2, \dots, s$ and $m \in \mathbb{Z}$,

$$\begin{aligned}
\mathbb{L}_{m,j} &:= \widetilde{\mathcal{U}}^{-1}\big(\overline{g_{m,j}(\cdot)}\, \mathrm{e}^{2\pi \mathrm{i} rm\cdot}\big) = \widetilde{\mathcal{U}}^{-1}\Big(\overline{\sum_{k\in\mathbb{Z}} \langle a, U^{k+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k\cdot}}\, \mathrm{e}^{2\pi \mathrm{i} rm\cdot}\Big) \\
&= \widetilde{\mathcal{U}}^{-1}\Big(\sum_{k\in\mathbb{Z}} \overline{\langle a, U^{k+\epsilon_{mj}} b_j\rangle}_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i}(rm-k)\cdot}\Big) = \widetilde{\mathcal{U}}^{-1}\Big(\sum_{k\in\mathbb{Z}} \overline{\langle a, U^{rm-k+\epsilon_{mj}} b_j\rangle}_{\mathcal{H}}\, \mathrm{e}^{2\pi \mathrm{i} k\cdot}\Big) \\
&= \big\{\overline{\langle a, U^{rm-k+\epsilon_{mj}} b_j\rangle}_{\mathcal{H}}\big\}_{k\in\mathbb{Z}}
\end{aligned}$$

The sequence $\{\mathbb{L}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\ell^2(\mathbb{Z})$. Indeed, assume that the error sequence $\boldsymbol{\epsilon} := \{\epsilon_{m,j}\}_{m\in\mathbb{Z};\, j=1,\dots,s}$ satisfies the hypothesis of Theorem 13, i.e.,

$$K_{\boldsymbol{\epsilon}} := \sum_{j=1}^{s} M_{a,b_j}(\gamma_j) N_{a,b_j}(\gamma_j) < \alpha_{\mathbb{G}}/r$$

where $\gamma_j := \sup_{m\in\mathbb{Z}} |\epsilon_{mj}|$ for each $j = 1, 2, \dots, s$. As a consequence, the sequence $\big\{\overline{g_{m,j}(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $L^2(0,1)$ with bounds

$$A_{\boldsymbol{\epsilon}} := \frac{\alpha_{\mathbb{G}}}{r}\left(1 - \sqrt{\frac{rK_{\boldsymbol{\epsilon}}}{\alpha_{\mathbb{G}}}}\right)^2 \quad \text{and} \quad B_{\boldsymbol{\epsilon}} := \frac{\beta_{\mathbb{G}}}{r}\left(1 - \sqrt{\frac{rK_{\boldsymbol{\epsilon}}}{\beta_{\mathbb{G}}}}\right)^2.$$

Since $\widetilde{\mathcal{U}}^{-1}$ is an isometry, the sequence $\{\mathbb{L}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is a frame for $\ell^2(\mathbb{Z})$ with the same bounds. Hence, the recovery of the element $x = \mathcal{T}_{U,a}\big(\widetilde{\mathcal{U}}\,\mathbb{F}\big) \in \mathcal{A}_a$ from the samples $\big\{\langle x, U^{rm+\epsilon_{mj}} b_j\rangle_{\mathcal{H}}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$ is reduced to recover $\mathbb{F} \in \ell^2(\mathbb{Z})$ from the sequence

$$\big\{\langle \mathbb{F}, \mathbb{L}_{m,j}\rangle_{\ell^2(\mathbb{Z})}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}\,.$$

In so doing, the classical frame algorithm reads (see, for instance, [7]): Let $\mathcal{S}$ be the frame operator in $\ell^2(\mathbb{Z})$ of $\{\mathbb{L}_{m,j}\}_{m\in\mathbb{Z};\, j=1,2,\dots,s}$, and define

$$\begin{aligned}
\mathbb{F}_0 :=& \frac{2\mathcal{S}}{A_{\boldsymbol{\epsilon}} + B_{\boldsymbol{\epsilon}}}\, \mathbb{F} = \frac{2}{A_{\boldsymbol{\epsilon}} + B_{\boldsymbol{\epsilon}}} \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \langle \mathbb{F}, \mathbb{L}_{m,j}\rangle\, \mathbb{L}_{m,j} \\
=& \frac{2}{A_{\boldsymbol{\epsilon}} + B_{\boldsymbol{\epsilon}}} \sum_{j=1}^{s} \sum_{m\in\mathbb{Z}} \langle x, U^{rm+\epsilon_{mj}} b_j\rangle\, \mathbb{L}_{m,j}\,,
\end{aligned}$$

and recursively,

$$\mathbb{F}_{k+1} = \mathbb{F}_k + \frac{2\mathcal{S}}{A_{\boldsymbol{\epsilon}} + B_{\boldsymbol{\epsilon}}}(\mathbb{F} - \mathbb{F}_k) \quad \text{for each } k \in \mathbb{N}\,.$$

Then, the sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}_a$ given by $x_n := \sum_{k\in\mathbb{Z}}(\mathbb{F}_n)_k\, U^k a$, satisfies

$$\begin{aligned}\|x - x_n\|_{\mathcal{H}} &\le \|\mathcal{T}_{U,a}\|\; \|\mathbb{F} - \mathbb{F}_k\|_{\ell^2(\mathbb{Z})} \le \|\mathcal{T}_{U,a}\|\, \gamma_{\boldsymbol{\epsilon}}^{k+1} \|\mathbb{F}\|_{\ell^2(\mathbb{Z})} \\ &\le \|\mathcal{T}_{U,a}\|\, \|\mathcal{T}_{U,a}^{-1}\|\, \gamma_{\boldsymbol{\epsilon}}^{k+1} \|x\|_{\mathcal{H}}\,,\end{aligned}$$

where $\gamma_{\boldsymbol{\epsilon}} := (B_{\boldsymbol{\epsilon}} - A_{\boldsymbol{\epsilon}})/(B_{\boldsymbol{\epsilon}} + A_{\boldsymbol{\epsilon}})$.

4 Some additional remarks

Given s vectors $b_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$ with $s \ge r$, we have proved in Section 2 that the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$ if and only if the constants $A_{\boldsymbol{\Psi}}$ and $B_{\boldsymbol{\Psi}}$ defined in (20) satisfy $0 < A_{\boldsymbol{\Psi}} \le B_{\boldsymbol{\Psi}} < \infty$. Furthermore, we have obtained a family of dual frames having the same form. As it was mentioned in the introduction, now we deal with the case that some $b_j \notin \mathcal{A}_a$.

- We have assumed in Theorems 5 and 7 that b_j belongs to $\mathcal{A}_a$ for each $j = 1, 2, \dots, s$ since we required the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ to be contained in $\mathcal{A}_a$. In case that some $b_j \notin \mathcal{A}_a$, the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is not necessarily contained in $\mathcal{A}_a$. However, a close look into the proof of Theorem 5 shows that whenever $0 < A_{\boldsymbol{\Psi}} \le B_{\boldsymbol{\Psi}} < \infty$, the inequalities

$$A_{\boldsymbol{\Psi}} \|x\|^2 \le \sum_{j=1}^{s} \sum_{k\in\mathbb{Z}} |\langle x, U^{rk} b_j\rangle|^2 \le B_{\boldsymbol{\Psi}} \|x\|^2 \quad \text{for all } x \in \mathcal{A}_a \tag{34}$$

  hold, and conversely. Hence, the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a pseudo-frame for $\mathcal{A}_a$ (see Refs. [17, 18]).

  Denoting by $P_{\mathcal{A}_a}$ the orthogonal projection onto $\mathcal{A}_a$, since for each $x \in \mathcal{A}_a$

$$\langle x, U^{rk} b_j\rangle = \big\langle x, P_{\mathcal{A}_a}\big(U^{rk} b_j\big)\big\rangle\,, \quad k \in \mathbb{Z} \text{ and } j = 1, 2, \dots, s\,,$$

  Theorems 5 and 7 can be reformulated in terms of $\{P_{\mathcal{A}_a}\big(U^{rk} b_j\big)\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ which is a sequence in $\mathcal{A}_a$.

- An analysis of the proof of Theorem 9 shows that, even if not all of the b_j belong to $\mathcal{A}_a$, there exist $c_j \in \mathcal{A}_a$, $j = 1, 2, \dots, s$, such that the sequence $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $\mathcal{A}_a$, and the expansion (22) holds for each $x \in \mathcal{A}_a$. Therefore, in case that some $b_j \notin \mathcal{A}_a$, the sequence $\{U^{rk} b_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ is a pseudo-dual frame of the frame $\{U^{rk} c_j\}_{k\in\mathbb{Z};\, j=1,2,\dots s}$ for $\mathcal{A}_a$ (see [17]).

- In Section 3.3, having in mind the isomorphism $\mathcal{T}_{U,a}$, for $x = \mathcal{T}_{U,a} F \in \mathcal{A}_a$ we have obtained the expressions

$$\langle x, U^{rm} b_j\rangle_{\mathcal{H}} = \big\langle F, \overline{g_j(w)}\, \mathrm{e}^{2\pi \mathrm{i} rmw}\big\rangle_{L^2(0,1)}\,, \quad \text{where } m \in \mathbb{Z} \text{ and } j = 1, 2, \dots, s\,.$$

Furthermore, we know that the sequence $\big\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ is a frame for $L^2(0,1)$ if and only if the constants $\alpha_{\mathbb{G}}$ and $\beta_{\mathbb{G}}$ given in (30) satisfy $0 < \alpha_{\mathbb{G}} \le \beta_{\mathbb{G}} < \infty$; besides, the optimal frame bounds for $\big\{\overline{g_j(w)}\,\mathrm{e}^{2\pi i r m w}\big\}_{m\in\mathbb{Z};\, j=1,2,\dots s}$ are $\alpha_{\mathbb{G}}/r$ and $\beta_{\mathbb{G}}/r$. Hence, we obtain that

$$\frac{\alpha_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}\|^{-2}\|x\|^2 \le \sum_{j=1}^{s}\sum_{k\in\mathbb{Z}}|\langle x, U^{rk}b_j\rangle|^2 \le \frac{\beta_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}^{-1}\|^2\|x\|^2\,, \quad x\in\mathcal{A}_a\,. \tag{35}$$

As a consequence, since we are dealing with optimal frame bounds, from (34) and (35) we derive the equalities:

$$A_{\mathbf{\Psi}} = \frac{\alpha_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}\|^{-2} \quad \text{and} \quad B_{\mathbf{\Psi}} = \frac{\beta_{\mathbb{G}}}{r}\|\mathcal{T}_{U,a}^{-1}\|^2\,.$$

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